

## BRAID GROUPS AND EUCLIDEAN SIMPLICES

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ABSTRACT. When Daan Krammer and Stephen Bigelow independently proved that braid groups are linear, they used the Lawrence-Krammer-Bigelow representation for generic values of its variables  $q$  and  $t$ . The  $t$  variable is closely connected to the traditional Garside structure of the braid group and plays a major role in Krammer's algebraic proof. The  $q$  variable, associated with the dual Garside structure of the braid group, has received less attention.

In this article we give a geometric interpretation of the  $q$  portion of the LKB representation in terms of an action of the braid group on the space of non-degenerate euclidean simplices. In our interpretation, braid group elements act by systematically reshaping (and relabeling) euclidean simplices. The reshaping associated to the simple elements in the dual Garside structure of the braid group are of an especially elementary type that we call relabeling and rescaling.

At the turn of the millenium three papers on the linearity of braid groups appeared in rapid succession and all three used what is now known as the Lawrence-Krammer-Bigelow or LKB representation [Kra00, Big01, Kra02]. Its two variables,  $q$  and  $t$ , are connected to two different Garside structures on the braid group. The  $t$  variable is closely connected to the traditional Garside structure of the braid group and plays a major role in Krammer's algebraic proof [Kra02]. The  $q$  variable is associated with the dual Garside structure and has received less attention. In this article, we introduce an elegant geometric interpretation of the  $q$  variable in the special case where  $t = 1$ ,  $q$  is real, and the matrices of the representation are written with respect to the original basis used by Krammer in [Kra00]. We call this special case the simplicial representation because of our first main result.

**Theorem A** (Braids reshape simplices). *The simplicial representation of the  $n$ -string braid group preserves the set of  $\binom{n}{2}$ -tuples of positive reals that represent the squared edge lengths of a nondegenerate euclidean simplex with  $n$  labeled vertices.*

Thus braid group elements can be viewed as acting on and systematically reshaping the space of all nondegenerate euclidean simplices. Moreover, the dual simple braids in the braid group reshape simplices in an extremely

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elementary way that we call relabeling and rescaling. In this language we prove the following result; for a more precise statement, see Section 3.

**Theorem B** (Dual simple braids relabel and rescale). *Under the simplicial representation of the braid group, each dual simple braid acts by relabeling the vertices and rescaling specific edges.*

The article is structured around the three distinct contexts that play a role in these results: discs, simplices and matrices. In Section 1 convexly punctured discs are used to define noncrossing partitions and dual simple braids. In Section 2 we describe various ways to systematically reshape euclidean simplices, including the type of reshaping that we call edge rescaling. In Section 3 we connect these two relatively elementary discussions with the explicit matrices of the simplicial representation of the braid group to establish our main results. Finally, Section 4 explains our motivation for pursuing this line of investigation and some ideas for future work.

## 1. DISCS

In this section metric discs with a finite number of labeled points are used to define the lattice of noncrossing partitions and the finite set of dual simple braids. We begin by recalling the notion of a convexly punctured disc.

**Definition 1.1** (Convexly punctured disc). Let  $\mathbf{D}_n$  be a topological disc in the euclidean plane with a distinguished  $n$ -element subset that we call its *punctures* or *vertices*. When the disc  $\mathbf{D}_n$  is a convex subset of  $\mathbb{R}^2$  and the convex hull of its  $n$  punctures is an  $n$ -gon (i.e. every puncture occurs as a vertex of the convex hull) then we say that  $\mathbf{D}_n$  is a *convexly punctured disc*. There is a natural cyclic ordering of the vertices corresponding to the clockwise orientation of the boundary cycle of the  $n$ -gon. A labeling of the vertices is said to be *standard* if it uses the set  $[n] := \{1, 2, \dots, n\}$  (or better yet  $\mathbb{Z}/n\mathbb{Z}$ ) and the vertices are labeled in the natural cyclic order. More generally, when the vertices  $p_i$  are bijectively labeled by elements  $i$  in a finite set  $A$ , we refer to the convexly punctured disc as  $\mathbf{D}_A$ .

The 2-elements subsets are of particular interest.

**Definition 1.2** (Edges). Let  $\mathbf{D}_n$  be a convexly punctured disc. For each two element subset  $\{i, j\} \subset [n]$ , the convex hull of the corresponding points  $p_i$  and  $p_j$  in  $\mathbf{D}_n$  is called an *edge* and denoted  $e_{i,j} = e_{j,i}$ , or even  $e_{ij}$  when the comma is not needed for clarity. When a *standard name* is needed we insist  $i < j$ . The number of edges is  $\binom{n}{2}$  and we consistently use  $N$  for this number throughout the article. For later use, it is also convenient to impose a *standard order* on the set of all  $N = \binom{n}{2}$  edges. We do so by lexicographically ordering them by their standard names. In  $\mathbf{D}_4$ , for example, the standard names of its  $6 = \binom{4}{2}$  edges in their standard order are  $e_{12}, e_{13}, e_{14}, e_{23}, e_{24}$  and  $e_{34}$ .

We also need words to describe the position of one edge relative to another.

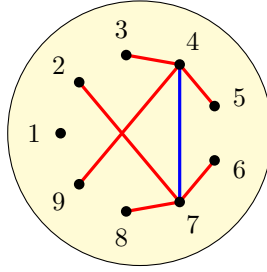


FIGURE 1. The edges  $e_{34}$ ,  $e_{49}$  and  $e_{67}$  are to the left of the edge  $e_{47}$  and the edges  $e_{27}$ ,  $e_{78}$  and  $e_{45}$  are to the right. This is because ordered pairs such as  $(e_{34}, e_{47})$  and  $(e_{67}, e_{47})$  are clockwise while the ordered pair  $(e_{27}, e_{47})$  is counterclockwise.

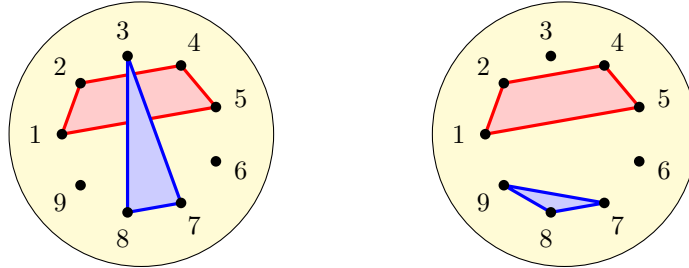
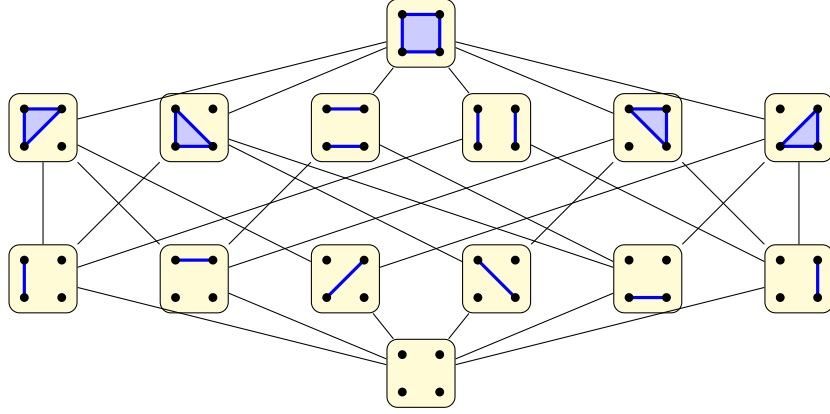


FIGURE 2. The subsets  $\{1, 2, 4, 5\}$  and  $\{3, 7, 8\}$  are crossing. The subsets  $\{1, 2, 4, 5\}$  and  $\{7, 8, 9\}$  are noncrossing.

**Definition 1.3** (Pairs of edges). Let  $(e_{ij}, e_{kl})$  be an ordered pair of edges in a convexly punctured disc  $\mathbf{D}_n$  and let  $B = \{i, j, k, l\}$ . If we restrict our attention to the convex subdisc  $\mathbf{D}_B$  (where  $\mathbf{D}_B$  is an  $\epsilon$ -neighborhood of the convex hull of the vertices indexed by the elements in  $B$ ) then there are exactly five distinct possible configurations. We call the possibilities crossing, noncrossing, identical, clockwise and counterclockwise. When all four endpoints are distinct (i.e. when  $|B| = 4$ ) these edges are either *crossing* or *noncrossing* depending on whether or not they intersect. At the other extreme, when  $e_{ij}$  and  $e_{kl}$  have both endpoints in common, they are *identical*. Finally, when these edges have exactly one endpoint in common, the convex hull of the three endpoints is a triangle and the edges occur as consecutive edges in its boundary cycle. We call this arrangement *clockwise* or *counterclockwise* depending on the orientation of the boundary which ensures that  $e_{kl}$  is the edge that occurs immediately after  $e_{ij}$ . More colloquially we say that  $e_{kl}$  is *to the right* (*left*) of  $e_{ij}$  and  $e_{ij}$  is *to the left* (*right*) of  $e_{kl}$  when the ordered pair  $(e_{ij}, e_{kl})$  is clockwise (counterclockwise). Examples are shown in Figure 1.

Noncrossing partitions are defined in a convexly punctured disc.

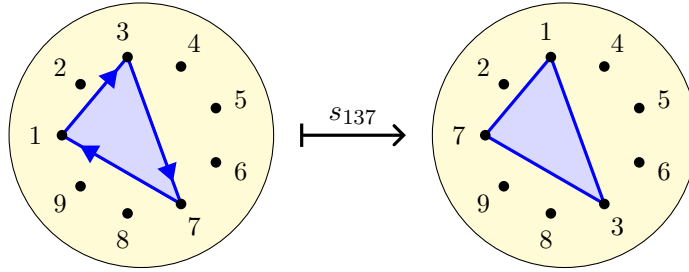
FIGURE 3. Noncrossing partition lattice  $NC_4$ .

**Definition 1.4** (Noncrossing partitions). Let  $\mathbf{D}_n$  be a convexly punctured disc. We say that two subsets  $B, B' \subset [n]$  are *noncrossing* when the convex hulls of the corresponding sets of vertices in  $\mathbf{D}_n$  are completely disjoint. See Figure 2. More generally, a partition  $\sigma$  of the set  $[n]$  is *noncrossing* when its blocks are pairwise noncrossing. Noncrossing partitions are usually ordered by refinement, so that  $\sigma < \tau$  if and only if each block of  $\sigma$  is contained in some block of  $\tau$ . Under this ordering, the set of all noncrossing partitions form a bounded graded lattice denoted  $NC_n$ . The poset  $NC_4$  is shown in Figure 3. The number of noncrossing partitions in  $NC_n$  is given by the  $n$ -th Catalan number  $C_n = \frac{1}{n+1} \binom{2n}{n}$ .

To each noncrossing partition there is a corresponding permutation.

**Definition 1.5** (Noncrossing permutations). To every subset  $B \subset [n]$  with at least two elements we associate a permutation in  $\text{SYM}_n$  obtained by linearly ordering the elements in  $B$ . More generally, we associate a permutation to each noncrossing partition by multiplying the permutations associated to each block and we call the result a *noncrossing permutation*. Since distinct blocks are disjoint, their permutations commute and the product is well-defined. Thus  $B = \{1, 3, 4\}$  becomes the permutation  $(1, 3, 4)$  and the partition  $\sigma = \{\{1, 3, 4\}, \{2\}, \{5, 6, 7, 8, 9\}\}$  becomes the permutation  $(1, 3, 4)(5, 6, 7, 8, 9)$ . We identify each noncrossing partition with its corresponding noncrossing permutation, using the same symbol for both. The permutation associated to the full set  $[n]$  is an important  $n$ -cycle that we call  $\delta$ .

As is well-known, the elements of the braid group can be identified with (equivalence classes of) motions of  $n$  distinct labeled points in a disc such as  $\mathbf{D}_n$ . The dual simple braids are a finite set of braids indexed by the noncrossing permutations as follows.


 FIGURE 4. The rotation  $s_{137}$ .

**Definition 1.6** (Rotations). The *dual Garside element*  $s_\delta$  of the  $n$ -string braid group is the motion where each labeled point in  $\mathbf{D}_n$  moves clockwise along the boundary of the convex hull of all  $n$  points to the next vertex. More generally, for each set  $B \subset \{1, \dots, n\}$ , let  $P_B$  be the convex hull of the vertices indexed by  $B$  and let  $\mathbf{D}_B$  be an  $\epsilon$ -neighborhood of  $P_B$ . The braid group element  $s_B$  is a similar motion restricted to the subdisc  $\mathbf{D}_B$ , i.e. the vertices in the subdisc move clockwise along one side of the polygon  $P_B$  to the next vertex, leaving all other vertices fixed. See Figure 4. When  $B$  has at most 1 element, the motion is trivial. When  $B$  has two elements, the points avoid collisions by passing on the left.

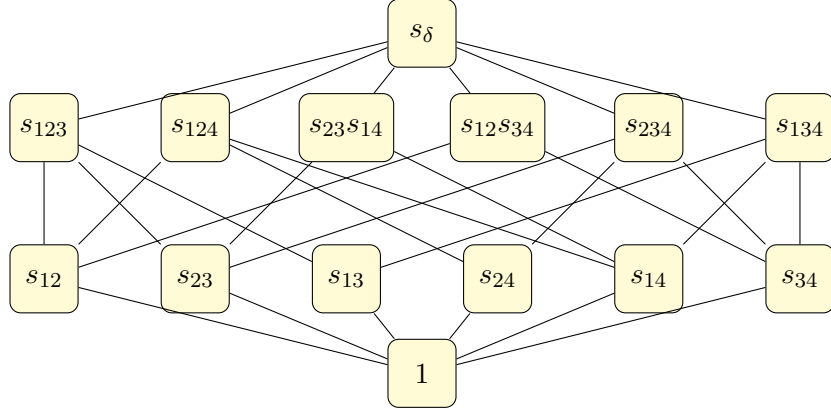
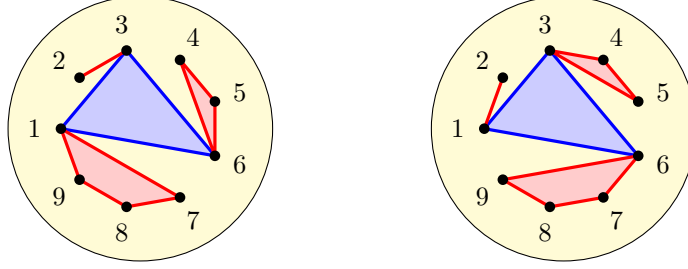
Rotations can be used to assign a braid to each noncrossing partition.

**Definition 1.7** (Dual simple braids). The *dual simple braids* are elements of the braid group in one-to-one correspondence with the set of noncrossing partitions  $NC_n$ . More precisely, for each noncrossing partition  $\sigma$ , we associate the product of the rotations corresponding to each of its blocks and call the result  $s_\sigma$ . Because rotations of noncrossing blocks take place in disjoint subdiscs they commute and the resulting element in the braid group is well-defined. Note that the noncrossing permutation  $\sigma$  is the permutation of the vertices induced by  $s_\sigma$ . The dual simple braids in  $\text{BRAID}_4$  written as products of rotations are shown in Figure 5.

It is useful to have specific names for four sets of dual simple braids.

**Definition 1.8** (Four sets of simple braids). The *standard generating set* for the braid group  $\text{BRAID}_n$  consists of the  $n - 1$  rotations of the form  $s_{ij}$  with  $1 \leq i < n$  and  $j = i + 1$ . The *dual generators* of  $\text{BRAID}_n$  are the set of all  $N = \binom{n}{2}$  rotations  $s_B$  where  $B$  has exactly two elements. The *rotations*  $s_B$  with  $|B| \neq 1$  form a third set, and the full set of all dual simple braids form a fourth set. We write  $\text{STD}_n \subset \text{GEN}_n \subset \text{ROT}_n \subset \text{SIMP}_n$  for these four nested sets, whose sizes are  $n - 1$ ,  $N$ ,  $2^n - n$  and  $C_n$  (the  $n$ -th Catalan number). When  $n = 4$ , these sets have 3, 6, 11 and 14 elements.

The multiplication in the symmetric group can be used to extract other information about noncrossing partitions.

FIGURE 5. The dual simple elements in  $\text{BRAID}_4$ .FIGURE 6.  $(23)(456)(1789)$  is the left complement of  $(136)$  in  $\text{SYM}_9$  and  $(12)(345)(6789)$  is its right complement.

**Definition 1.9** (Multiplication). For consistency with our latter conventions we view permutations as functions and thus we multiply them from *right to left*. For example, the product  $(1, 2, 3) \cdot (3, 4, 5)$  is  $(1, 2, 3, 4, 5)$ . More generally, if  $B$ ,  $\{i\}$  and  $C$  are pairwise disjoint subsets of  $[n]$  such that there is a place to start reading the boundary cycle of the convex hull of the points corresponding to the elements in  $B \cup \{i\} \cup C$  so that, reading clockwise, one encounters all of the vertices indexed by elements in  $B$ , followed by  $p_i$ , followed by all of the vertices indexed by the elements in  $C$ , then  $s_{Bi}s_{iC} = s_{BiC}$ . Here we follow the conventions of [MM09], removing parentheses from singletons, and using juxtaposition to indicate union.

The reader should be careful to note that our multiplication convention differs from much of the literature on the braid groups where multiplication is from left to right. Thus extra vigilance is required. The general multiplication rule given above, for example, is stated in a slightly different form in [MM09]. The multiplication can be used to define left and right complements of noncrossing permutations.

**Definition 1.10** (Complements). Let  $\sigma$  be a noncrossing permutation in  $\text{SYM}_n$  and recall that  $\delta$  is the  $n$ -cycle  $(1, 2, \dots, n)$ . The *left complement* of

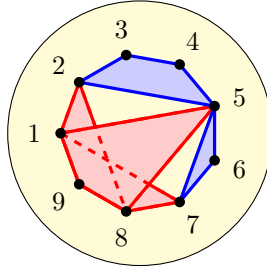


FIGURE 7. If  $\sigma_1 = (2, 3, 4, 5)$  and  $\sigma_2 = (5, 6, 7)$ , then the permutations  $\sigma_3$ ,  $\sigma_4$  and  $\sigma_5$  defined in Definition 1.12 are  $\sigma_3 = (1, 7, 8, 9)$ ,  $\sigma_4 = (1, 5, 8, 9)$  and  $\sigma_5 = (1, 2, 8, 9)$ .

$\sigma$  is the unique element  $\sigma'$  such that  $\sigma'\sigma = \delta$  and its *right complement* is the unique element  $\sigma''$  such that  $\sigma\sigma'' = \delta$ . We denote these permutations by  $\sigma' = \text{lc}(\sigma)$  and  $\sigma'' = \text{rc}(\sigma)$ . The permutation  $\text{lc}(\sigma)$  is always also a noncrossing permutation and, in fact, the edges in its blocks are precisely those that are to the left or noncrossing with respect to each of the edges in the blocks of  $\sigma$ . Similarly  $\text{rc}(\sigma)$  is a noncrossing permutation whose blocks are formed by the edges that are to the right or noncrossing with respect to each edge in a block of  $\sigma$ . See Figure 6.

The following observation is not crucial to our results, but we sometimes use this language.

**Remark 1.11** (Hypertrees). A *hypergraph* is a generalization of a graph where its *hyperedges* are allowed to span more than two vertices, and a *hypertree* is the natural generalization of a tree. As can be seen in Figure 6, the blocks of the noncrossing partition associated to a dual simple element and the blocks of one of its complements together form the hyperedges of a planar hypertree.

The Hasse diagram of the noncrossing partition lattice can actually be viewed as a portion of the Cayley graph of  $\text{SYM}_n$  with respect the generating set of all transpositions, specifically the portion between the identity and the element  $\delta$ . This way of looking at the noncrossing partition lattice combined with the fact that the set of transpositions in  $\text{SYM}_n$  is closed under conjugation, helps to explain why factorizations of  $\delta$  as a product of noncrossing permutations are so flexible. The only aspect of this flexibility that we need here is the following.

**Definition 1.12** (Five permutations). If  $\sigma_1$  and  $\sigma_2$  are permutations in  $\text{SYM}_n$  such that  $\sigma_1$ ,  $\sigma_2$  and their product  $\sigma_1\sigma_2$  are all three noncrossing, then there exist noncrossing permutations  $\sigma_3$ ,  $\sigma_4$  and  $\sigma_5$  such that  $\delta = \sigma_1\sigma_2\sigma_3 = \sigma_1\sigma_4\sigma_2 = \sigma_5\sigma_1\sigma_2$ . The permutations  $\sigma_5$  and  $\sigma_3$  are simply the left and right complements of the product  $\sigma_1\sigma_2$ , while  $\sigma_4$  is obtained by conjugation. An example is shown in Figure 7.

## 2. SIMPLICES

In this section we discuss the geometry of euclidean simplices with  $n$  labeled vertices and, in particular, how this geometry changes under certain carefully controlled deformations. As in [BM], we start by distinguishing between points and vectors.

**Definition 2.1** (Points). Let  $V$  be an  $(n-1)$ -dimensional *real vector space* with a fixed positive definite inner product but no fixed basis and let  $E$  be an  $(n-1)$ -dimensional *euclidean space*, which may be defined as a set with a fixed simply-transitive action of the additive group of  $V$ . The structure of  $E$  is essentially that of  $V$  but the location of the origin has been forgotten. The elements of  $V$  are *vectors*, the elements of  $E$  are *points*, and we write  $\langle u, v \rangle$  for the inner product of vectors  $u$  and  $v$ . Two points  $p$  and  $p'$  determine a line segment  $e$  called an *edge* and  $p$  and  $p'$  are its *endpoints*. By the simply-transitive action of  $V$  on  $E$ , they also determine two vectors: the unique vector  $v$  that sends  $p$  to  $p'$  and the vector  $-v$  which sends  $p'$  to  $p$ . The pair  $\pm v$  is a *lax vector*. When the points involved are labeled, say  $p_i$  and  $p_j$ , we write  $e_{ij} = e_{ji}$  for the edge they span, and  $v = v_{ij}$  and  $-v = v_{ji}$  for the two vectors they determine. The *norm* of a vector  $v$  is  $\langle v, v \rangle$ , which is also the square of the length of the corresponding edge  $e$ . We write  $\text{NORM} : V \rightarrow \mathbb{R}$  for the norm map. Note that the norm of a lax vector is well-defined since  $\text{NORM}(v) = \text{NORM}(-v)$  and the norm of an edge is the norm of the lax vector determined by its endpoints. When the points involved are labeled we write  $a_{ij}$  for  $\text{NORM}(v_{ij})$ .

**Definition 2.2** (Simplices). A set  $\{p_i\}$  of  $n$  labeled points in  $E$  is in *general position* if this set is not contained in any proper affine subspace of  $E$ , and the convex hull of such a set is a *labeled euclidean simplex*  $\Delta$  of dimension  $(n-1)$ . For any such labeled euclidean simplex  $\Delta$  we use subsets of punctures in the convexly punctured disc  $\mathbf{D}_n$  to describe various simplicial faces of  $\Delta$  via their vertex labelings. For example, the three blocks of the left complement of  $s_{136}$  shown in Figure 6 correspond to an edge, a triangle and a tetrahedron in any 8-dimensional simplex  $\Delta$  with 9 labeled vertices.

We are primarily interested in the isometry class of a labeled euclidean simplex  $\Delta$  and this is completely determined by the ordered list of the norms of its edges.

**Definition 2.3** (Edge norm vectors). Let  $\Delta$  be a labeled euclidean simplex with  $n$  vertices. The *edge norm vector* of  $\Delta$  is a column vector  $\mathbf{v}$  of the  $N = \binom{n}{2}$  positive real numbers  $a_{ij}$  which are the norms of its edges  $e_{ij}$ , listed in the standard lexicographic order of the edges as discussed in Definition 1.2.

Edge norm vectors characterize isometry classes of labeled euclidean simplices and as a result when we reshape a labeled euclidean simplex, these changes to its geometry are captured by the modifications that occur in its edge norm vector. The well-known formula  $2\langle u, v \rangle = \text{NORM}(u+v) -$



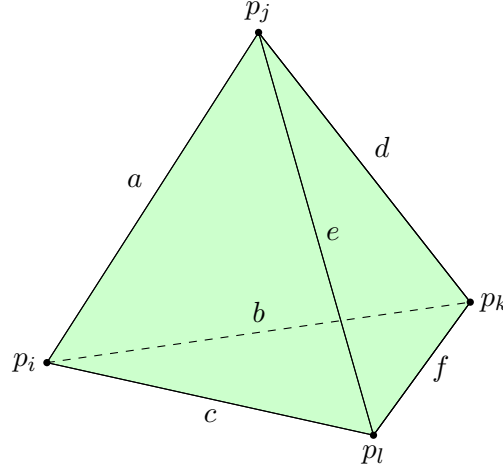


FIGURE 8. Tetrahedron determined by 4 points, edges labeled by norm.

$\text{NORM}(u) - \text{NORM}(v)$  shows that inner products of vectors can be calculated in terms of their norms, but we need a slightly more general formula that computes the inner product of two vectors determined by four possibly distinct points in a euclidean space  $E$ .

**Proposition 2.4** (Inner products and norms). *If  $p_i, p_j, p_k$  and  $p_l$  are four not necessarily distinct points in a euclidean space  $E$ , then the inner product  $2\langle v_{ij}, v_{kl} \rangle = a_{il} + a_{jk} - a_{ik} - a_{jl}$ .*

*Proof.* To improve readability, we write  $a, b, c, d, e$  and  $f$  for the norms  $a_{ij}, a_{ik}, a_{il}, a_{jk}, a_{jl}$  and  $a_{kl}$ , respectively. The situation under discussion is shown in Figure 8. Expanding the norms of  $v_{ik} = v_{ij} + v_{jk}$  and  $v_{jl} = v_{jk} + v_{kl}$  produces the identities  $b = a + d + 2\langle v_{ij}, v_{jk} \rangle$  and  $e = d + f + 2\langle v_{jk}, v_{kl} \rangle$ . Expanding the norm of  $v_{il} = v_{ij} + v_{jk} + v_{kl}$  produces  $c = a + d + f + 2\langle v_{ij}, v_{jk} \rangle + 2\langle v_{jk}, v_{kl} \rangle + 2\langle v_{ij}, v_{kl} \rangle$ . Thus  $2\langle v_{ij}, v_{jk} \rangle = b - a - d$ ,  $2\langle v_{jk}, v_{kl} \rangle = e - d - f$ , and  $2\langle v_{ij}, v_{kl} \rangle = c - a - d - f - (b - a - d) - (e - d - f) = c + d - b - e$ .  $\square$

The following definition identifies a class of geometric reshaping of labeled euclidean simplices that are particularly elegant and easy to describe. We call them edge rescalings and, as far as we are aware, they have not been previously discussed in the literature.

**Definition 2.5** (Edge rescaling). Let  $\Delta$  and  $\Delta'$  be two labeled euclidean simplices with  $n$  vertices situated in a common euclidean space. We say that an edge  $e_{ij}$  in  $\Delta$  is merely *rescaled* if it and the corresponding edge  $e'_{ij}$  in  $\Delta'$  point in the same direction. More generally, we say that  $\Delta'$  is an *edge rescaling* of  $\Delta$  if there exist enough pairs of corresponding edges pointing in the same direction (but with possibly different lengths) to form a basis for the vector space out of these common direction vectors.

**Remark 2.6** (Spanning trees). Let  $\Delta'$  be a labeled euclidean simplex which is an edge rescaling of  $\Delta$ . By definition there are sufficiently many edges that are merely rescaled to form a basis out of the corresponding vectors and a minimal set of rescaled edges in the 1-skeleton of  $\Delta$  would form a spanning tree in this complete graph. There might, however, be more than one such spanning tree of merely rescaled edges. When two edges in  $\Delta$  share a common endpoint and are both rescaled by the same scale factor, the triangle they span, and in particular the third edge in that triangle, is also rescaled by the same scale factor. Thus any two of these three edges could be included in the spanning tree. In fact, it would be more canonical to identify the *maximal* simplicial faces that are merely rescaled. For each scale factor, there would be a partition of the vertices into maximal subsimplices rescaled by that factor and the blocks in all of these partitions together would form a spanning hypertree in the sense of Remark 1.11. The full variety of spanning trees which satisfy the edge rescaling definition are selected from the edges inside the blocks of such a canonical spanning hypertree.

**Definition 2.7** (Edge rescaling maps). One thing to note is that for every spanning tree  $T$  in the 1-skeleton of a labeled euclidean simplex  $\Delta$  and for every set of positive real scale factors for these edges, there does exist a rescaled simplex  $\Delta'$  in which these edges are rescaled by these factors since the rescaled tree  $T'$  formed by assembling the rescaled edges as before tells us how the vertices should be arranged. In particular, the rescaling of  $\Delta$  only depends on the set of merely rescaled edges and the scale factors used to rescale them. Thus there is a well-defined *edge rescaling map*  $R$  from the space of labeled euclidean simplices to itself which, based only on such data, rescales each  $\Delta$  in the set to a new simplex  $\Delta'$ . Such a map  $R$  is clearly invertible since rescaling the same edges by the multiplicative inverse of each scale factor returns each  $\Delta'$  to  $\Delta$ . The edge rescaling maps we are primarily interested in are those where every scale factor is 1 or  $q$ . We call these  *$q$ -rescalings* and we introduce a special notation for them. Let  $R = R_\tau^\sigma$  denote the  $q$ -rescaling where the blocks of the partition  $\sigma$  index the subsimplices rescaled by  $q$  and the blocks of the partition  $\tau$  index the subsimplices which are fixed, i.e. rescaled by a factor of 1.

The next proposition shows that the effect of  $R$  is encoded in a matrix.

**Proposition 2.8** (Edge rescaling matrices). *The effect of an edge rescaling map  $R$  on an edge norm vector  $\mathbf{v}$  for a labeled euclidean simplex  $\Delta$  is captured by an  $N$  by  $N$  matrix  $M$  whose entries only depend on the map  $R$  and not on the vector  $\mathbf{v}$  or the simplex  $\Delta$ . In particular,  $R(\mathbf{v}) = M \cdot \mathbf{v}$  for all  $\mathbf{v}$  and  $\Delta$ .*

*Proof.* Let  $T$  be a spanning set of edges that are rescaled by  $R$ . For each edge  $e_{ij}$  we can use paths in the spanning tree  $T$  to find a linear combination of vectors associated with rescaled edges whose sum is  $v_{ij}$ . The new vector  $v'_{ij}$ , by definition, is a similar sum where the vectors in the sum are rescaled

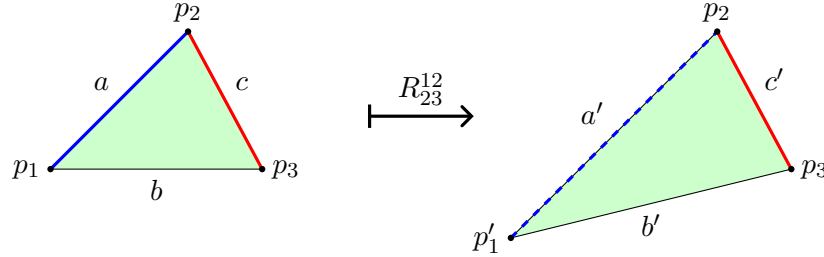


FIGURE 9. An edge reshaping  $R = R_{23}^{12}$  that rescales  $e_{12}$  by a factor of  $q$  and fixes  $e_{23}$  (i.e. rescales  $e_{23}$  by a factor of 1).

according to the scale factors of  $R$ . Thus the norm of  $v'_{ij}$  can be expanded as a linear combination of inner products of vectors whose edges belong to the tree  $T$  and the coefficients of this linear combination are independent of the original edge norms. Next, by Proposition 2.4, the inner products can be rewritten as linear combinations of the original edge norms. Substituting these in produces a formula for each new edge norm  $a'_{ij}$  as a linear combination of the old edge norms in  $\mathbf{v}$  with coefficients that are independent of  $\mathbf{v}$ . The matrix  $M$  is formed by collecting these coefficients.  $\square$

For simplicity we use the same symbol  $R$  to denote both the edge rescaling map and the edge rescaling matrix that was called  $M$  in the proposition. As an explicit example of this process, consider the  $q$ -rescaling of a triangle shown in Figure 9.

**Proposition 2.9** (Edge rescaling a triangle). *If  $\Delta$  is a labeled euclidean triangle and  $\Delta'$  is the labeled euclidean triangle obtained by the edge rescaling  $R = R_{23}^{12}$ , then the edge norms of  $\Delta'$  can be computed from the edge norms of  $\Delta$  as follows:  $a'_{12} = q^2 a_{12}$ ,  $a'_{13} = (q^2 - q)a_{12} + qa_{13} + (1 - q)a_{23}$ , and  $a'_{23} = a_{23}$ . In particular,  $R$  has the effect of multiplying the edge norm vector of  $\Delta$  by a matrix with entries in  $\mathbb{Z}[q]$ .*

$$(1) \quad \mathbf{v}' = \begin{bmatrix} a'_{12} \\ a'_{13} \\ a'_{23} \end{bmatrix} = \begin{bmatrix} q^2 & 0 & 0 \\ q^2 - q & q & 1 - q \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_{12} \\ a_{13} \\ a_{23} \end{bmatrix} = R \cdot \begin{bmatrix} a_{12} \\ a_{13} \\ a_{23} \end{bmatrix} = R \cdot \mathbf{v}$$

*Proof.* For clarity let  $a$ ,  $b$  and  $c$  be the edge norms  $a_{12}$ ,  $a_{13}$  and  $a_{23}$  in  $\Delta$  and add primes for the corresponding edge norms in  $\Delta'$ . In the original triangle we have  $a = \langle v_{12}, v_{12} \rangle$ ,  $c = \langle v_{23}, v_{23} \rangle$  and  $b = \langle v_{13}, v_{13} \rangle = \langle v_{12} + v_{23}, v_{12} + v_{23} \rangle = a + 2\langle v_{12}, v_{23} \rangle + c$ . Thus  $2\langle v_{12}, v_{23} \rangle = b - a - c$ . In the new triangle we have  $c' = \langle v_{23}, v_{23} \rangle = c$ ,  $a' = \langle q \cdot v_{12}, q \cdot v_{12} \rangle = q^2 a$ , and  $b' = \langle q \cdot v_{12} + v_{23}, q \cdot v_{12} + v_{23} \rangle = q^2 a + 2q\langle v_{12}, v_{23} \rangle + c = q^2 a + q(b - a - c) + c = (q^2 - q)a + qb + (1 - q)c$  as required.  $\square$

Many of the properties of the matrix  $R = R_{23}^{12}$  extend to all  $q$ -rescalings.

**Proposition 2.10** (Quadratic matrices). *Let  $R = R_\tau^\sigma$  be a  $q$ -rescaling of a labeled euclidean simplex  $\Delta$  with  $n$  vertices. The effect of  $R$  on the edge*

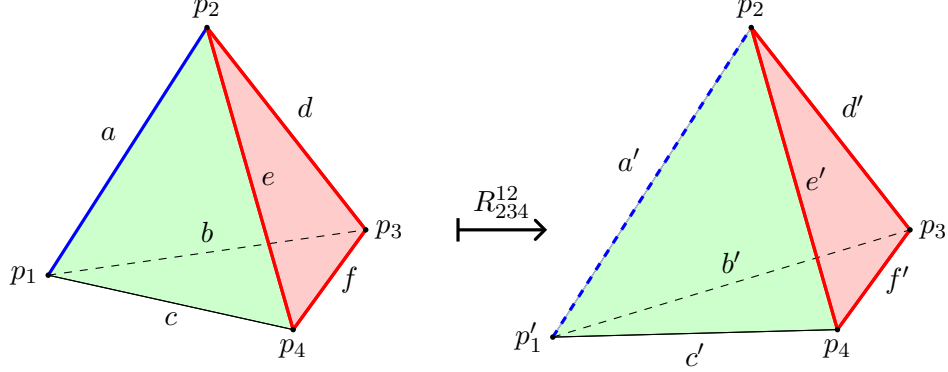


FIGURE 10. The edge rescaling  $R_{234}^{12}$  which fixes the triangle  $\Delta_{234}$  and rescales edge  $e_{12}$  by a factor of  $q$ .

norm vector  $\mathbf{v}$  of  $\Delta$  is to multiply from the left by an  $N$  by  $N$  matrix with entries in  $\mathbb{Z}[q]$  of degree at most 2.

*Proof.* The proof is the same as that of Proposition 2.8 but with the additional observation that the coefficients are at most quadratic polynomials in  $q$  since there is at most one  $q$  coming from each side of the inner product.  $\square$

Our second example is very similar to the first.

**Example 2.11** (Rescaling a tetrahedron). Consider the edge rescaling  $R_{234}^{12}$  as shown in Figure 10 and note that we can compute all of the new edge norms using Proposition 2.9. For clarity we write  $a$  through  $f$  for  $a_{12}$  through  $a_{34}$  in lexicographic order. The effect of the map  $R = R_{234}^{12}$  is as follows:

$$(2) \quad R \cdot \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} q^2 a \\ (q^2 - q)a + qb + (1 - q)d \\ (q^2 - q)a + qc + (1 - q)e \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a' \\ b' \\ c' \\ d' \\ e' \\ f' \end{bmatrix}$$

Thus the matrix that encodes the rescaling  $R$  is

$$(3) \quad R = \begin{bmatrix} q^2 & 0 & 0 & 0 & 0 & 0 \\ q^2 - q & q & 0 & 1 - q & 0 & 0 \\ q^2 - q & 0 & q & 0 & 1 - q & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

When the simplices are higher-dimensional, additional notation is needed.

**Definition 2.12** (Row descriptions). Since the new edge norms are determined by the rows of the matrix  $R = R_r^\sigma$ , we introduce a way to describe

these rows. Recall that  $e_{ij}$  denotes an edge in the punctured disc  $\mathbf{D}_n$  and an edge in a labeled euclidean simplex  $\Delta$ . We now add a third interpretation: as an element of the canonical basis of the vector space  $\mathbb{R}^N$  containing the edge norm vectors. Using this interpretation of  $e_{ij}$  as basis vectors, the second row of the matrix for  $R = R_{23}^{12}$  as given in Proposition 2.9 is the row vector  $(q^2 - q, q, 1 - q)$  or, equivalently, it is the linear combination  $(q^2 - q)e_{12} + qe_{13} + (1 - q)e_{23}$ . To select the second row one would multiply the matrix  $R$  on the right by the row vector  $(0, 1, 0)$  which is just the vector  $e_{13}$ . In other words,  $(e_{13})R = (q^2 - q)e_{12} + qe_{13} + (1 - q)e_{23}$ .

**Remark 2.13** (Left and right). The linear combination that describes the image of a basis vector acted on by a rescaling matrix  $R$  from the right looks very similar to the corresponding entry in the edge norm vector when acted on by  $R$  from the left precisely because both are essentially encoding the entries in one row of  $R$ . Nevertheless, this switch between left and right has the potential to be slightly confusing.

It turns out that the linear combination that describes the  $e_{kl}$  row of the matrix  $R_{\text{rc}(ij)}^{ij}$  (or  $R_{\text{lc}(ij)}^{ij}$ ) only depends on the geometric relationship between the edges  $e_{ij}$  and  $e_{kl}$  in the punctured disc  $\mathbf{D}_n$ . Thus, in our row descriptions of these matrices the final column uses the language of Definition 1.3 to describe how  $e_{kl}$  is situated relative to  $e_{ij}$ .

**Example 2.14** (Rescaling a boundary edge). In this example we give explicit row descriptions for the  $q$ -rescalings which stretch a single boundary edge while fixing either its left or its right complement. We start with the row description of the matrix  $R = R_{\text{rc}(12)}^{12}$ .

$$(4) \quad (e_{kl})R = \begin{cases} q^2 e_{kl} & \text{identical } (k = 1, l = 2) \\ e_{kl} & \text{noncrossing } (k, l > 2) \\ e_{kl} & \text{to the right } (k = 2) \\ (q^2 - q)e_{12} + qe_{kl} + (1 - q)e_{2l} & \text{to the left } (k = 1) \end{cases}$$

This is a natural generalization of the triangular and tetrahedral examples in the new notation. The row description of  $R = R_{\text{rc}(ij)}^{ij}$  with  $j = i + 1 \pmod n$  is only slightly more complicated. We write  $e_{\text{new}}$  for the third edge of the triangle when  $e_{ij}$  and  $e_{kl}$  have exactly one endpoint in common.

$$(5) \quad (e_{kl})R = \begin{cases} q^2 e_{kl} & \text{identical} \\ e_{kl} & \text{noncrossing} \\ e_{kl} & \text{to the right} \\ (q^2 - q)e_{ij} + qe_{kl} + (1 - q)e_{\text{new}} & \text{to the left} \end{cases}$$

Switching from the right complement to the left complement causes only very minor changes. The row description of the matrix  $R = R_{\text{lc}(12)}^{12}$  is as

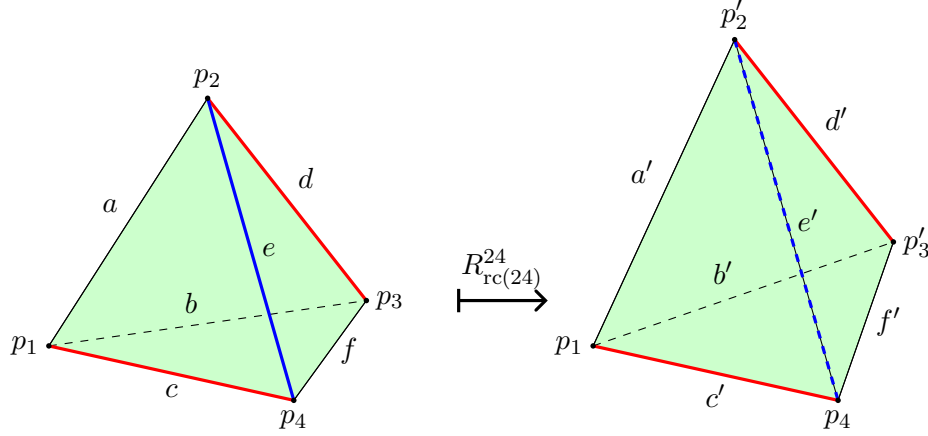


FIGURE 11. The edge rescaling  $R_{rc(24)}^{24}$  which rescales the edge  $e_{24}$  while fixing the edges  $e_{23}$  and  $e_{14}$ .

follows.

$$(6) \quad (e_{kl})R = \begin{cases} q^2 e_{kl} & \text{identical } (k=1, l=2) \\ e_{kl} & \text{noncrossing } (k, l > 2) \\ e_{kl} & \text{to the left } (k=1) \\ (q^2 - q)e_{12} + qe_{kl} + (1-q)e_{1l} & \text{to the right } (k=2) \end{cases}$$

And finally, we list the row description of  $R_{lc(ij)}^{ij}$  with  $j = i+1 \pmod n$ , with the same convention that  $e_{\text{new}}$  denotes the third edge of the triangle when  $e_{ij}$  and  $e_{kl}$  have exactly one endpoint in common.

$$(7) \quad (e_{kl})R = \begin{cases} q^2 e_{kl} & \text{identical} \\ e_{kl} & \text{noncrossing} \\ e_{kl} & \text{to the left} \\ (q^2 - q)e_{ij} + qe_{kl} + (1-q)e_{\text{new}} & \text{to the right} \end{cases}$$

In order to extend Example 2.14 to more general situations, we need one more computation.

**Proposition 2.15** (Diagonal edges). *Let  $\Delta$  be a labeled euclidean tetrahedron. If  $\Delta'$  is the labeled euclidean tetrahedron obtained by the edge rescaling  $R = R_{rc(24)}^{24}$ , then the new edge norm  $a'_{13}$  can be computed from the edge norms of  $\Delta$  as follows:*

$$a'_{13} = a_{13} + (q-1)^2 a_{24} + (q-1)(a_{14} + a_{23}) + (1-q)(a_{12} + a_{34})$$

*Proof.* Since  $v_{13} = v_{12} + v_{24} + v_{43}$ , we have  $v'_{13} = v_{12} + qv_{24} + v_{43}$ . Expanding  $a'_{13} = \langle v'_{13}, v'_{13} \rangle$  we find that  $a'_{13}$  is equal to  $a_{12} + q^2 a_{24} + a_{34} + 2q \langle v_{12}, v_{24} \rangle + 2q \langle v_{24}, v_{43} \rangle + 2 \langle v_{12}, v_{34} \rangle$ . Using Proposition 2.4 we find that  $\langle v_{12}, v_{24} \rangle = a_{14} - a_{12} - a_{24}$ ,  $\langle v_{24}, v_{43} \rangle = a_{23} - a_{24} - a_{34}$  and  $\langle v_{12}, v_{43} \rangle = a_{13} + a_{24} - a_{14} - a_{23}$ . Substituting and simplifying yields the result.  $\square$

**Example 2.16** (Rescaling a diagonal edge). The row description for the rescaling matrix  $R_{\text{rc}(ij)}^{ij}$  is essentially identical to the one listed in Example 2.14. In particular, the formulas for the cases where  $e_{kl}$  is identical to, noncrossing, to the left or to the right of  $e_{ij}$  are the same as before. The final geometric configuration that is possible when  $e_{ij}$  is not a boundary edge is that  $e_{ij}$  and  $e_{kl}$  might be crossing. When this happens  $(e_{kl})R_{\text{rc}(ij)}^{ij}$  can be computed using Proposition 2.15. If the clockwise ordering of  $i, j, k$  and  $l$  is  $(k, i, l, j)$  then the answer is  $e_{kl} + (q - 1)^2 e_{ij} + (q - 1)e_{kj} + (q - 1)e_{il} + (1 - q)e_{ki} + (1 - q)e_{lj}$ . A more intrinsic geometric description would use the convex hull of  $\{p_i, p_j, p_k, p_l\}$  in  $\mathbf{D}_n$ . The answer obtained is  $e_{kl}$  plus  $(q - 1)^2 e_{ij}$  plus  $(q - 1)$  times the two boundary edges of the convex hull which are simultaneously to the right of  $e_{ij}$  and to the left of  $e_{kl}$  plus  $(1 - q)$  times the two boundary edges which are simultaneously to the right of  $e_{ij}$  and to the left of  $e_{kl}$ . A row description for  $R_{\text{lc}(ij)}^{ij}$  for arbitrary  $i$  and  $j$  can be computed in a similar fashion.

### 3. MATRICES

In this section, we define three explicit representations of the braid group and then we prove our main results. The first representation, and the most complicated, is the Lawrence-Krammer-Bigelow or LKB representation.

**Definition 3.1** (LKB representation). Let  $q$  and  $t$  be nonzero positive real numbers, let  $\mathcal{E}$  be the set  $\{e_{ij}\}$  with  $1 \leq i < j \leq n$  of size  $N = \binom{n}{2}$  and let  $\mathbb{R}^N$  be the  $N = \binom{n}{2}$ -dimensional real vector space with  $\mathcal{E}$  as its ordered basis. The *LKB representation* of the braid group is the map  $\rho : \text{BRAID}_n \rightarrow GL_N(\mathbb{R})$  defined by the following action (from the right) of the standard braid group generators  $s_{ij}$  (with  $j = i + 1$  and  $1 \leq i < n$ ) on elements of  $\mathcal{E}$ .

$$(8) \quad (e_{kl})\rho(s_{ij}) = \begin{cases} tq^2 e_{kl} & i = k, j = l \\ e_{kl} & i, j \notin \{k, l\} \\ e_{jl} & i = k, j < l \\ e_{kj} & i = l \\ t(q^2 - q)e_{ij} + qe_{ki} + (1 - q)e_{kl} & k < i, j = l \\ (q^2 - q)e_{ij} + qe_{il} + (1 - q)e_{kl} & j = k \end{cases}$$

We make two remarks about this definition.

**Remark 3.2** (Left/right actions). In the literature, this action is written as an action from the left but our alteration only has the effect of transposing the relevant matrices. Our choice is dictated by our desire to match up (a simplified version of) this representation with the obviously very similar edge rescaling matrices discussed in the previous section.

**Remark 3.3** (The  $t$  variable and its sign). The  $t$  variable depends on the linear ordering of the vertices, it is associated the standard presentation of the braid group, and its presence obscures the fundamentally cyclically symmetric nature of the dependence on  $q$ . To highlight this cyclic symmetry,

we shall consider the specialization with  $t = 1$  below. We should also note, however, that there are inconsistencies in the literature regarding the sign of  $t$ . The variable  $t$  in [Kra00] corresponds to  $-t$  in [Big01] (with an additional sign correction in [Big03]) and [IW]. We have written the LKB representation using Krammer's sign convention. If we had followed Bigelow's we would be setting  $t$  equal to  $-1$ .

In light of our first main theorem, we call the simplified LKB representation the *simplicial representation* of the braid group.

**Definition 3.4** (Simplicial representation). The *simplicial representation* of the braid group is the specialization of the LKB representation with  $t$  set equal to 1. Concretely, let  $q$  be a nonzero positive real number, let  $\mathcal{E}$  be the set  $\{e_{ij}\}$  with  $1 \leq i < j \leq n$  in lexicographic order and let  $\mathbb{R}^N$  be the  $N = \binom{n}{2}$ -dimensional real vector space with  $\mathcal{E}$  as its ordered basis. The *simplicial representation* of the braid group is defined by the following action (from the right) of the standard braid group generators  $s_{ij}$  (with  $j = i + 1$  and  $1 \leq i < n$ ) on elements of  $\mathcal{E}$ . We write  $S_\sigma$  for the matrix that represents  $s_\sigma$  with respect to the ordered basis  $\mathcal{E}$  and we have introduced the notation  $e_{\text{new}}$  to denote the third side of the triangle when  $e_{ij}$  and  $e_{kl}$  have exactly one endpoint in common as in the previous section.

$$(9) \quad (e_{kl})S_{ij} = \begin{cases} q^2 e_{kl} & i = k, j = l \\ e_{kl} & i, j \notin \{k, l\} \\ e_{\text{new}} & i = k, j < l \\ e_{\text{new}} & i = l \\ (q^2 - q)e_{ij} + qe_{\text{new}} + (1 - q)e_{kl} & k < i, j = l \\ (q^2 - q)e_{ij} + qe_{\text{new}} + (1 - q)e_{kl} & j = k \end{cases}$$

Now that the  $t$  variable has been eliminated, some of the rows are identical and they can be rewritten more elegantly using the language of Definition 1.3.

$$(10) \quad (e_{kl})S_{ij} = \begin{cases} q^2 e_{kl} & \text{identical} \\ e_{kl} & \text{noncrossing} \\ e_{\text{new}} & \text{to the left} \\ (q^2 - q)e_{ij} + qe_{\text{new}} + (1 - q)e_{kl} & \text{to the right} \end{cases}$$

Our third representation is obtained by also eliminating the  $q$  variable.

**Definition 3.5** (Permutation representation). The *permutation representation* of the braid group that we are interested in is the one obtained from the simplicial representation by setting  $q = 1$  (or both  $t = q = 1$  in the LKB representation). This encodes the permutation of the edges induced by the corresponding permutation of the vertices. We write  $P_\sigma$  for the matrix corresponding to  $s_\sigma$ . Its row description is as follows.

$$(11) \quad (e_{kl})P_{ij} = \begin{cases} e_{kl} & \text{identical, crossing or noncrossing} \\ e_{\text{new}} & \text{to the left or right} \end{cases}$$



It should be clear that the simplicial representation matrix  $S_{ij}$  is very closely connected, but not quite identical to the rescaling matrix  $R_{\text{rc}(ij)}^{ij}$  given in Example 2.14. The difference is the permutation matrix  $P_{ij}$ .

**Example 3.6** (Geometry of  $S_{12}$ ). The matrices corresponding to the first standard generator  $s_{12}$  of the four string braid group in the simplicial representation and the permutation representation are as follows:

$$(12) \quad S_{12} = \begin{bmatrix} q^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ q^2 - q & q & 0 & 1 - q & 0 & 0 \\ q^2 - q & 0 & q & 0 & 1 - q & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad P_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It is now straightforward to check that  $S_{12} = P_{12}R_{234}^{12} = R_{134}^{12}P_{12}$ . The matrix  $R_{234}^{12} = R_{\text{rc}(12)}^{12}$  is listed explicitly in Example 2.11 and the row description of  $R_{134}^{12} = R_{\text{lc}(12)}^{12}$  is given in Example 2.14.

Before proceeding to the proofs of our main results, we introduce one final definition which makes our precise results easier to state.

**Definition 3.7** (Relabeling and rescaling). Let  $\sigma$  be a noncrossing permutation in  $\text{SYM}_n$  and let  $S_\sigma$  be the explicit matrix representing  $s_\sigma$  under the simplicial representation. We say that  $S_\sigma$  *relabels and rescales* if  $S_\sigma = P_\sigma R_{\text{rc}(\sigma)}^\sigma = R_{\text{lc}(\sigma)}^\sigma P_\sigma$ , where  $\text{lc}(\sigma)$  and  $\text{rc}(\sigma)$  are the left/right complements of  $\sigma$ .

For clarity we also say what this definition means geometrically. To say that the matrix  $S_\sigma$  relabels and rescales means that the effect it has on labeled euclidean simplices (keeping in mind that we multiply from right to left) is to first rescale the edges (fixing the length and direction of the edges in the right complement of  $\sigma$  while multiplying the lengths of the edges in  $\sigma$  by a factor of  $q$ ) followed by the edge relabeling induced by the way  $\sigma$  permutes the vertices. Alternatively, the edge relabeling  $P_\sigma$  can be performed first (i.e. on the right), in which case it is the edges of the left complement of  $\sigma$  whose length and direction are fixed while the lengths of the edges in  $\sigma$  are multiplied by a factor of  $q$ . In this language, Example 3.6 shows that the matrix  $S_{12}$  in the simplicial representation of  $\text{BRAID}_4$  relabels and rescales. The next proposition shows that the standard generators in all of the braid groups share this property.

**Proposition 3.8** (Standard generators). *For every standard generator  $s_{ij}$  of the braid group  $\text{BRAID}_n$ , the corresponding matrix  $S_{ij}$  in the simplicial representation relabels and rescales.*

*Proof.* This is essentially immediate at this point once we compare the row description of  $S_{ij}$  in Definition 3.4 with the row descriptions of  $R_{\text{rc}(ij)}^{ij}$  and

$R_{lc(ij)}^{ij}$  in Example 2.14 and note that multiplying by  $P_{ij}$  on the left switches the rows to the left of  $e_{ij}$  with the rows to the right of  $e_{ij}$  (which has the effect of switching which edge is denoted  $e_{kl}$  and which is  $e_{\text{new}}$ ), while multiplying by  $P_{ij}$  on the right permutes columns and thus the subscripts on the  $e$ 's that occur in the various terms of the row descriptions.  $\square$

From Proposition 3.8 we deduce our first result.

**Theorem A** (Braids reshape simplices). *The simplicial representation of the  $n$ -string braid group preserves the set of  $\binom{n}{2}$ -tuples of positive reals that represent the squared edge lengths of a nondegenerate euclidean simplex with  $n$  labeled vertices.*

*Proof.* First note that it suffices to prove that this holds for some generating set of  $\text{BRAID}_n$ . Next, both vertex relabelings and edge rescalings clearly preserve the set of  $\binom{n}{2}$ -tuples that describe the squared edge lengths of a nondegenerate euclidean simplex with  $n$  labeled vertices, so Proposition 3.8 completes the proof.  $\square$

After we proved Theorem A, we discovered that it had already been established (in a different language) in the dissertation of Arkadius Kalka [Kal07]. Since our proof is more geometric in nature and seemingly simpler than his, we believe that our proof is of independent interest. Also, he does not appear to have established anything resembling our Theorem B. Once we know that some dual simple braids relabel and rescale, it is straightforward to show that their products (at least those which remain dual simple braids) do so as well.

**Proposition 3.9** (Products). *Let  $\sigma_1$  and  $\sigma_2$  be noncrossing permutations in  $\text{SYM}_n$  such that  $s_{\sigma_1}$ ,  $s_{\sigma_2}$  and  $s_{\sigma_1\sigma_2}$  are dual simple braids. If both  $S_{\sigma_1}$  and  $S_{\sigma_2}$  relabel and rescale, then  $S_{\sigma_1\sigma_2}$  relabels and rescales.*

*Proof.* One consequence of the fact that  $s_{\sigma_1}$ ,  $s_{\sigma_2}$  and  $s_{\sigma_1\sigma_2}$  are dual simple braids is that there are noncrossing permutations  $\sigma_3$ ,  $\sigma_4$  and  $\sigma_5$  such that  $\delta = \sigma_1\sigma_2\sigma_3 = \sigma_1\sigma_4\sigma_2 = \sigma_5\sigma_1\sigma_2$  (Definition 1.12). For  $S_{\sigma_1\sigma_2} = S_{\sigma_1}S_{\sigma_2}$  we have the following equalities.

$$\begin{aligned} S_{\sigma_1}S_{\sigma_2} &= (P_{\sigma_1}R_{\sigma_4\sigma_2}^{\sigma_1})(R_{\sigma_1\sigma_4}^{\sigma_2}P_{\sigma_2}) \\ &= P_{\sigma_1}R_{\sigma_4}^{\sigma_1\sigma_2}P_{\sigma_2} \\ &= P_{\sigma_1}P_{\sigma_2}R_{\sigma_3}^{\sigma_1\sigma_2} \\ &= R_{\sigma_5}^{\sigma_1\sigma_2}P_{\sigma_1}P_{\sigma_2} \end{aligned}$$

The first line uses the hypotheses on  $S_{\sigma_1}$  and  $S_{\sigma_2}$ . The second combines the two edge scalings into a single rescaling. In particular, the rescaling  $R_{\sigma_4\sigma_2}^{\sigma_1}$  rescales the edges in  $\sigma_1$  by  $q$  and fixes the edges in  $\sigma_4$  and  $\sigma_2$ , as well as the rest of the edges in the product  $\sigma_4\sigma_2$ . Similarly, the rescaling  $R_{\sigma_1\sigma_4}^{\sigma_2}$  fixes the edges in  $\sigma_1$  and  $\sigma_4$  and rescales those in  $\sigma_2$  by a factor of  $q$ . Thus, in the product of these two edge rescalings, the edges in  $\sigma_1$  and  $\sigma_2$  are rescaled

by  $q$  and those in  $\sigma_4$  are fixed. The third and fourth lines simply conjugate the points involved. Since  $P_{\sigma_1}P_{\sigma_2} = P_{\sigma_1\sigma_2}$  and  $\sigma_5$  and  $\sigma_3$  are the left/right complements of  $\sigma_1\sigma_2$ , this completes the proof.  $\square$

Note that the same equalities used in the proof, slightly rearranged, would show that if any two of  $S_{\sigma_1}$ ,  $S_{\sigma_2}$  and  $S_{\sigma_1\sigma_2}$  relabel and rescale then so does the third. Propositions 3.8 and 3.9 are not quite enough to prove our second main result because not all dual simple braids are products of standard generators. We need to extend Proposition 3.8 to the full set of dual generators.

**Proposition 3.10** (Dual generators). *For every dual generator  $s_{ij} \in \text{GEN}_n$  of the braid group  $\text{BRAID}_n$ , the corresponding matrix  $S_{ij}$  in the simplicial representation relabels and rescales.*

*Proof.* An explicit description of the matrix for  $s_{ij}$  under the LKB representation is given in Krammer's earlier paper [Kra00]. If we set  $t = 1$  in that description, we find that the matrix  $S_{ij}$  has the exact same description as it does when  $j = i + 1$  as given in Definition 3.4 except that a new case must be added that gives the result of  $(e_{kl})S_{ij}$  when  $e_{ij}$  and  $e_{kl}$  cross. This simplification of the answer listed in [Kra00] agrees with the corresponding row of  $P_{ij}R_{\text{rc}(ij)}^{ij}$ , which is the same as the corresponding row of  $R_{\text{lc}(ij)}^{ij}P_{ij}$  obtained by combining Example 2.16 and Definition 3.5.  $\square$

Our second main result is an immediate corollary.

**Theorem B** (Dual simple braids relabel and rescale). *Under the simplicial representation of the braid group, each dual simple braid relabels and rescales. Concretely, for each  $\sigma \in NC_n$ , we have  $S_\sigma = P_\sigma R_{\text{rc}(\sigma)}^\sigma = R_{\text{lc}(\sigma)}^\sigma P_\sigma$ .*

*Proof.* Proposition 3.10 shows that the assertion is true for each dual generator. Then Proposition 3.9 and a simple induction extends this fact to all of the dual simple braids.  $\square$

#### 4. FINAL REMARKS

In this final section we make a few remarks about the origins of these results and some promising directions for further investigation.

To explain how we stumbled upon this point of view, we first need to review the differences between the three papers by Krammer and Bigelow establishing the linearity of the braid groups. In his earlier article Daan Krammer used a ping-pong type argument on the dual Garside structure of the 4 string braid group to establish that the LKB representation representation is faithful for suitably generic values of  $q$  and  $t$  [Kra00]. Next Stephen Bigelow replaced Krammer's algebraic approach with a more topological one and succeeded in showing that these representations are faithful for all  $n$  [Big01]. In his later article Krammer used an alternative version of his original approach that also succeeded in establishing linearity for every

$n$  [Kra02]. The main difference between the two papers by Krammer is that the first uses the dual Garside presentation of the braid groups (indexed by the  $q$  variable and closely related to the Birman-Ko-Lee presentation introduced in [BKL98]) while the second reverts to the standard presentation of the braid group (indexed by the  $t$  variable).

Shortly after these articles appeared, their results were extended to prove linearity results for various other Artin groups, all based more or less on the approach used in Krammer's second article [Kra02]. François Digne extended linearity to the Artin groups of crystallographic type [Dig03] and Arjeh Cohen and David Wales proved linearity for all spherical Artin groups [CW02]. Finally Luis Paris generalized these results further by proving that the Artin *monoid* is linear for all Artin groups [Par02]. In fact, all of these proofs establish the linearity of the positive monoid. When the Artin group is spherical, the Artin group is the group of fractions of the positive monoid and thus linearity of the positive monoid implies linearity for the group.

For general Artin groups it is known that the positive monoid generated by the standard minimal generating set is not large enough for this implication to hold. On the other hand, the positive monoid generated by the larger dual generating set might have this property. This led us to attempt to generalize Krammer's initial argument using the dual generating set. If one could find a proof of linearity for all the braid groups focused on the  $q$  variable as in [Kra00], then there is a chance that it might generalize as in the papers by Digne, Cohen and Wales and Paris to eventually produce linearity results for non-spherical Artin groups. This was our initial motivation.

We first wrote code in `sage` to investigate the properties of the LKB matrices and found, experimentally, nice ways to decompose them and we began to isolate the changes that each factor was making. The interpretation of the simplified version as modifications of edge norms of simplices was one of the final steps in our evolving understanding of these representations.

We conclude this article with three directions for additional research.

**Problem 4.1** (Linearity). The simplified simplicial representation is known not to be faithful for large  $n$  because it is also known as the symmetric tensor square of the Burau representation (which is known to not be faithful for  $n \geq 5$  [Big99]). Thus, establishing a new  $q$  based proof of braid group linearity requires the variable  $t$  to remain generic. One project is to find a way to extend the geometric interpretation given here so that the  $t$  variable can remain generic and then to use this extended interpretation to give an alternative proof of braid group linearity focused on the  $q$  variable. In particular, one should try to show directly that the dual positive monoid generated by full dual generating set acts faithfully using a ping-pong type argument similar to the one in the original Krammer article on  $\text{BRAID}_4$  [Kra00].

**Problem 4.2** (Dual Garside length). The  $q$  variable in the LKB representation has received very little attention since Krammer's earlier paper

primarily because it was the  $t$  variable which was the focus for the more general proof. Recently, however, Tetsuya Ito and Bert Wiest posted an article that proves one of the facts originally conjectured by Krammer in [Kra00], namely, that the highest power of  $q$  in the LKB representation of a dual positive braid is twice its dual Garside length [IW]. It seems quite likely that one could give an alternative and elementary proof of their results using the geometric understanding of the  $q$  variable introduced in this article.

**Problem 4.3** (Spherical Artin groups). The set of labeled euclidean simplices, with dilated simplices identified, is one of the standard parameterizations of the higher rank symmetric space  $SL(V)/SO(V)$  and the simplicial representation appears to act on this space by isometries. Once this action is made explicit, it should be possible to define a similar construction and to give a similar interpretation for all of the spherical Artin groups once the focus on labeled euclidean simplices is replaced by linear transformations of root systems.

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